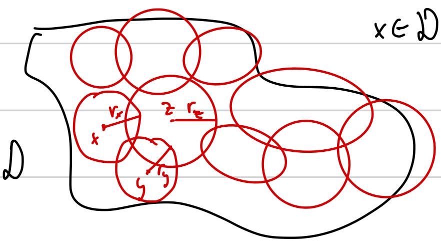
Measure Theory with Ergodic Horizons
Lecture 28

Hardy-Littlewood Maximal Theorem. Let $h \in L^1(\mathbb{R}^d, \lambda)$ and $\alpha > 0$. Then
$$\alpha \cdot \lambda(D_\alpha(A^* h)) \leq 3^d \|h\|_1.$$
In fact, $\alpha \cdot \lambda(D_\alpha(\bar{A} h)) \leq 3^d \|h\|_1$, where $\bar{A} h := \sup_{r \leq 1} A_r |h|$ is the so-called Hardy-Littlewood maximal function.

Proof. Denote $D := D_\alpha(\bar{A} h) := \{x \in \mathbb{R}^d : \sup_{r \leq 1} A_r |h| > \alpha\}$ and note that for each



$x \in D$ there is $0 < r_x \leq 1$ such that $A_{r_x} |h| > \alpha$, equivalently:
$$\int_{B_{r_x}(x)} |h| \, d\lambda \geq \alpha \cdot \lambda(B_{r_x}(x)). \qquad (+)$$
We want to deduce from this that $\int_D |h| \, d\lambda \geq \alpha \, \lambda(D)$, up to a constant multiple.

If there were only countably many such balls and they were pairwise disjoint, then by ctbl additivity of the integral and (+), we would get $\int_D |h| \, d\lambda \geq \alpha \, \lambda(D)$. Thus, the following tiling lemma is critical:

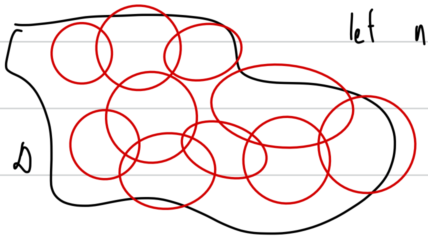
Vitali Covering Lemma. Let $D \subseteq \mathbb{R}^d$ be any λ-measurable set of positive measure and let $\mathcal{C}$ be a cover of D with balls (in, say, d_∞ metric). For each $0 < a < \lambda(D)$, there is a finite disjoint subcollection $\mathcal{C}_0 \subseteq \mathcal{C}$ such that
$$\lambda\Big(\bigsqcup_{B \in \mathcal{C}_0} B\Big) \geq \frac{1}{3^d} \cdot a.$$

Granted this lemma, we finish the proof of the Hardy-Littlewood theorem as follows. For each $a < \lambda(\mathcal{D})$ let $\mathcal{C} := \{B_{r_x}(x) : x \in \mathcal{D}\}$ and get a finite disjoint $\mathcal{C}_0 \subseteq \mathcal{C}$ from the Vitali covering lemma, so $\lambda(C_0) \geq \frac{1}{3^d} \cdot a$, where $C_0 := \bigsqcup_{B \in \mathcal{C}_0} B$. Then:

$$\|h\|_1 \geq \int_{C_0} |h| \, d\lambda = \sum_{B \in \mathcal{C}_0} \int_B |h| \, d\lambda \overset{\text{by } (\dagger)}{\geq} \sum_{B \in \mathcal{C}_0} \alpha \, \lambda(B) = \alpha \cdot \lambda(C_0) \geq \alpha \cdot \frac{1}{3^d} \cdot a \to \alpha \cdot \frac{1}{3^d} \cdot \lambda(\mathcal{D}).$$

$\square$

<u>Proof of Vitali covering.</u> By choosing a large enough ball B, we have that $\lambda(\mathcal{D} \cap B) > a$, so by replacing $\mathcal{D}$ with $\mathcal{D} \cap B$, we may assume that $\lambda(\mathcal{D}) < \infty$. Now tightness applies and gives a compact $K \subseteq \mathcal{D}$ with $\lambda(K) > a$, so replacing $\mathcal{D}$ with K, we may assume that $\mathcal{D}$ is compact. Since $\mathcal{C}$ is a cover of $\mathcal{D}$, there is a finite subcover $\mathcal{C}' \subseteq \mathcal{C}$. Enumerate $\mathcal{C}'$ by decreasing diameter of balls: $B_1, \dots, B_n$. We greedily define a disjoint subsequence of this balls:



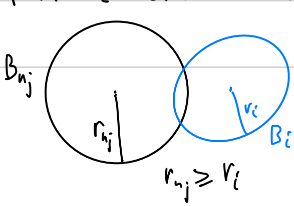
let $n_1 := 1$, then $n_2 := \min\{i : B_i \cap B_{n_1} = \emptyset\}$,

then $n_3 := \min\{i : B_i \cap (B_{n_1} \cup B_{n_2}) = \emptyset\}$

$\vdots$

$n_k := \min\{i : B_i \cap \bigcup_{j<k} B_{n_j} = \emptyset\}$.

For a ball B let $\tilde{B}$ denote the ball with the same center but 3 times the radius. Then we claim that $\bigcup_{j=1}^{k} \tilde{B}_{n_j} \supseteq \bigcup_{i=1}^{n} B_i$. Indeed, for each $i \in \{1, \dots, n\}$, there is $n_j \leq i$ such that $B_{n_j} \cap B_i \neq \emptyset$, so $\tilde{B}_{n_j} \supseteq B_i$ because



the radius of B_{n_j} is $\geq$ the radius of B_i. But then

$$\lambda\left(\bigsqcup_{j=1}^{k} B_{n_j}\right) = \sum_{j=1}^{k} \lambda(B_{n_j}) = \frac{1}{3^d} \sum_{j=1}^{k} \lambda(\tilde{B}_{n_j}) \geq \frac{1}{3^d} \lambda\left(\bigcup_{j=1}^{k} \tilde{B}_{n_j}\right) \geq \frac{1}{3^d} \lambda\left(\bigcup_{i=1}^{n} B_i\right) \geq \frac{1}{3^d} \cdot \lambda(O). \quad \square$$

Technical Strengthening of Lebesgue Differentiation. For any $f \in L^1_{loc}(\mathbb{R}^d, \lambda)$ and for a.e. $x_0 \in \mathbb{R}^d$,

$$\lim_{r \to 0} \frac{1}{\lambda(B_r(x_0))} \int_{B_r(x_0)} |f(y) - f(x_0)| \, d\lambda(y) = 0 \qquad (\#)$$

Proof. We know for a fixed x_0, applying the Lebesgue diff. thm. to $|f(x) - f(x_0)|$, we get that for a conull set $C_{x_0} \subseteq \mathbb{R}^d$, for every $x \in C_{x_0}$, we have:

$$\lim_{r \to 0} \frac{1}{\lambda(B_r(x))} \int_{B_r(x)} |f(y) - f(x_0)| \, d\lambda(y) = |f(x) - f(x_0)|.$$

If $x_0 \in C_{x_0}$, then we get $(\#)$, but x_0 may not be in C_{x_0}. We would like to take $\bigcap_{x_0 \in \mathbb{R}^d} C_{x_0}$ and "hope" that it is still a conull set. To make this intuitive idea work, we use $\mathbb{Q}$.

For each $q \in \mathbb{Q}$, we apply the Leb. diff. thm. to $f - q$ and obtain a conull $C_q \subseteq \mathbb{R}^d$ such that for all $x \in C_q$, we have

$$\lim_{r \to 0} \frac{1}{\lambda(B_r(x))} \int_{B_r(x)} |f(y) - q| \, d\lambda(y) = |f(x) - q|.$$

Now $C := \bigcap_{q \in \mathbb{Q}} C_q$ is indeed conull and we verify (#) for each $x_0 \in C$.
Fix $x_0 \in C$ and $\varepsilon > 0$. Let $q \in \mathbb{Q}$ with $|f(x_0) - q| < \frac{\varepsilon}{2}$. Denote $c := f(x_0)$.
Then for each $r > 0$,

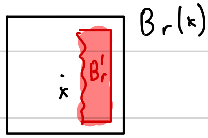
$$A_r|f-c|(x_0) = \frac{1}{\lambda(B_r(x_0))} \int_{B_r(x_0)} |f-c| \, d\lambda \overset{\text{triangle ineq.}}{\leq} \frac{1}{\lambda(B_r(x_0))} \int_{B_r(x_0)} |f-q| \, d\lambda + \frac{1}{\lambda(B_r(x_0))} \int_{B_r(x_0)} |q-c| \, d\lambda$$

$$\xrightarrow[r \to 0]{} |f(x_0) - q| + |q-c| < \varepsilon. \qquad \square$$

Def. For $x \in \mathbb{R}^d$, we say that a family $\{B'_r\}_{r>0}$ of λ-measurable sets *shrinks nicely to x* if $\exists\, p \in (0,1)$ such that $\forall r > 0$

(i) $B'_r \subseteq B_r(x)$

(ii) $\lambda(\tilde{B}_r) \geq p \cdot \lambda(B_r(x))$.



Cor (another strengthening of Leb. diff. thm). For each $f \in L^1_{loc}(\mathbb{R}^d)$ and a.e. $x \in \mathbb{R}^d$,

$$\lim_{r \to 0} \frac{1}{\lambda(B'_r(x))} \cdot \int_{B'_r(x)} |f(y) - f(x)| \, d\lambda = 0$$

for each family $\{B'_r(x)\}_{r>0}$ that shrinks nicely to x.

Proof. We know that for a.e. $x \in \mathbb{R}^d$,

$$\lim_{r \to 0} \frac{1}{\lambda(B_r(x))} \cdot \int_{B_r(x)} |f(y) - f(x)| \, d\lambda = 0.$$

Let $\{B'_r(x)\}_{r>0}$ be a family shrinking nicely to x, i.e. there is a $p \in (0,1)$

such that for all $r>0$, $B'_r(x) \subseteq B_r(x)$ and $\lambda(B'_r(x)) \geq p \cdot \lambda(B_r(x))$.

Then $$\frac{1}{\lambda(B'_r(x))} \cdot \int_{B'_r(x)} |f(y)-f(x)|\,d\lambda \leq \frac{1}{p\cdot\lambda(B_r(x))} \cdot \int_{B_r(x)} |f(y)-f(x)|\,d\lambda \xrightarrow[r\to 0]{} 0.$$ $\square$

Lebesgue Differentiation of singular measures.

We proved that for a loc. finite Borel measure μ on $\mathbb{R}^d$, if $\mu \ll \lambda$, then $\lim_{r\to 0} \frac{\mu(B_r(x))}{\lambda(B_r(x))} = \frac{d\mu}{d\lambda}(x)$ λ-a.e. What if $\mu \perp \lambda$? For example, let μ be the pushforward of Bernoulli$(\frac{1}{2})$ measure from $2^{\mathbb{N}}$ to the standard Cantor set $C \subseteq [0,1]$. Then for each $x \in \mathbb{R}\setminus C$, we have $\mu(B_r(x)) = 0$ for small enough $r>0$ because $B_r(x) \subseteq \mathbb{R}\setminus C$. Thus, $\lim_{r\to 0} \frac{\mu(B_r(x))}{\lambda(B_r(x))} = 0$ for all $x \in \mathbb{R}\setminus C$, hence for λ-a.e. $x \in \mathbb{R}$.

This is true more generally:

Thm. Let μ be a loc. finite Borel measure on $\mathbb{R}^d$. If $\mu \perp \lambda$, then for λ-a.e. $x \in \mathbb{R}^d$ and any family $\{B'_r(x)\}_{r>0}$ that shrinks nicely to x,
$$\lim_{r\to 0} \frac{\mu(B'_r(x))}{\lambda(B'_r(x))} = 0.$$

Proof. It's enough to prove for balls in class because, if $p \in (0,1)$ is the shrinking nicely constant, then

$$\frac{\mu(B'_r(x))}{\lambda(B_r(x))} \leq \frac{\mu(B_r(x))}{p\cdot\lambda(B_r(x))} \to 0 \quad \text{as } r \to 0.$$

By $\mu \perp \lambda$, we have $\mathbb{R}^d = X_\lambda \sqcup X_\mu$, where X_λ is λ-conull and X_μ is μ-conull. It is enough to show that the sets

$$Z_\alpha := \left\{ x \in X_\lambda : \limsup_{r\to 0} \frac{\mu(B_r(x))}{\lambda(B_r(x))} > \alpha \right\}$$

are λ-null because $\bigcup_{\alpha>0} Z_\alpha = \bigcup_{n\in\mathbb{N}^+} Z_{1/n}$ is then λ-null. Fix $\varepsilon > 0$ and show $\lambda(Z_\alpha) < \varepsilon$.

By the regularity of μ, since $\mu(Z_\alpha) = 0$, there is an open set $U \supseteq Z_\alpha$ with $\mu(U) < \varepsilon\cdot\alpha/3^d$. Now for each $x \in Z_\alpha$ there is $r_x > 0$ such that $B_{r_x}(x) \subseteq U$ and

$$\frac{\mu(B_{r_x}(x))}{\lambda(B_{r_x}(x))} > \alpha \qquad (*)$$

by the definition of limsup. Then $U \supseteq \bigcup_{x\in Z_\alpha} B_{r_x}(x) \supseteq Z_\alpha$, so we may replace U with $\bigcup_{x\in Z_\alpha} B_{r_x}(x)$ and assume that $U = \bigcup_{x\in Z_\alpha} B_{r_x}(x)$. In particular, $\mathcal{C} := \{B_{r_x}(x) : x \in Z_\alpha\}$ is a cover of U so for each positive $a < \lambda(U)$, the Vitali covering lemma gives a finite disjoint $\mathcal{C}_0 \subseteq \mathcal{C}$ with $\lambda(\bigsqcup_{B\in\mathcal{C}_0} B) \geq a/3^d$. Then

$$a \leq 3^d \lambda\Big(\bigsqcup_{B\in\mathcal{C}_0} B\Big) = 3^d \sum_{B\in\mathcal{C}_0} \lambda(B) \overset{(*)}{<} \frac{3^d}{\alpha} \sum_{B\in\mathcal{C}_0} \mu(B_{r_x}(x)) = \frac{3^d}{\alpha}\cdot \mu\Big(\bigsqcup_{B\in\mathcal{C}_0} B\Big) \leq \frac{3^d}{\alpha}\cdot\mu(U) < \varepsilon.$$

Since $a < \lambda(U)$ is arbitrary, we get $\lambda(U) < \varepsilon$, so $\lambda(Z_\alpha) < \varepsilon$, as desired. $\square$